§4. Growth of Fourier coefs and L-functions of modular forms.

We start with the growth of F-coefs in the case Eisenstein series

<u>Lemma 4-1</u> Let $f = E_k(z) = \sum a_n q^n$

Then $\exists$ constants A, B s.t

$$A n^{k-1} \leq |a_n| \leq B n^{k-1}$$

<u>Proof.</u> Recall $a_n = A\,\sigma_{k-1}(n)$ for some A

$$A\sigma_{k-1}(n) = A \sum_{d|n} d^{k-1} \geq A n^{k-1}$$

On the other hand

$$\frac{|a_n|}{n^{k-1}} = A \sum_{d|n} \frac{d^{k-1}}{n^{k-1}} = A \sum_{d|n} \frac{1}{d^{k-1}} \leq A \sum_{n=1}^{\infty} \frac{1}{n^{k-1}}$$
$$= A\zeta(k-1) = B$$

~~Hence~~

$$|a_n| < B n^{k-1} \quad \blacksquare$$

For cusp forms we can prove

<u>Thm 4-2</u> Let $f \in S_k(\Gamma)$, $f = \sum_{n=1}^{\infty} a_n q^n$

Then $|a_n| \leq C n^{k/2}$ $\forall n$ for some constant C depending only on f.

Pf f is a cusp form, hence $a_0 = 0$ and

$$f(z) = a_1 q + a_2 q^2 + \dots$$
$$= q(a_1 + a_2 q + \dots)$$

Hence

$$|f(z)| = O(|q|) = O(e^{-2\pi y}) \qquad |q| = |e^{2\pi i x}| \, |e^{-2\pi y}|$$

Hence $|f(z)| y^{k/2} \to 0$ as $y \to \infty$.

Now $|f(z)| y^{k/2}$ is continuous and hence bounded on the compact part of the fund. domain $\overline{F}_T := \{ z \in F \mid \operatorname{Im} z \leq T \}$

Since $|f(z)| y^{k/2} \to 0$ as $y \to \infty$, it is also bounded on for $\operatorname{Im} z > T$. Hence in fact $|f(z)| y^{k/2}$ is bounded on all of F.

Now note $|f(\gamma z)| (\operatorname{Im} \gamma z)^{k/2} = |cz+d|^k |f(z)| \dfrac{y^{k/2}}{|cz+d|^k}$

$$= |f(z)| y^{k/2}$$

ie $|f(z)| y^{k/2}$ is invariant under Γ.

That means in fact $|f(z)| y^{k/2}$ is bounded $\forall z \in \mathbb{H}$.

We've shown that if $f \in S_k$ then $|f(z)| < M y^{-k/2}$ $\forall z \in \mathbb{H}$ for some M.

Now for fixed y, we have seen before that

$$a_n e^{-2\pi n y} = \int_0^1 f(x+iy) e^{-2\pi i n x} \, dx .$$

Hence

$$|a_n| e^{-2\pi n y} < \int_0^1 |f(x+iy)| \, dx < M y^{-k/2}$$

ie $|a_n| < M y^{-k/2} e^{2\pi n y}$. choose $y = 1/n$

to get $|a_n| < C n^{k/2}$ with $C = M e^{2\pi}$

<u>Rmk.</u> Note for a cusp form f we've seen

$y^{k/2} f(x+iy)$ is bounded

Conversly if $y^{k/2} f(x+iy)$ is bounded

then $a_0 = 0$.

ie f is a cusp form $\Leftrightarrow$ $y^{k/2} f(x+iy)$ is bdd.

ie $a_0 = 0$.

<u>Cor 4.3.</u> If $f = \sum c_n q^n \in M_k \setminus S_k$ then $O(n^{k-1})$

<u>Proof</u> $f = \sum c_n q^n \in M_k \Rightarrow f = \lambda E_k + g$ for some

$\lambda \in \mathbb{C}$, and $g = \sum b_n q^n \in S_k$

$a_n(E_k) = O(n^{k-1})$, $b_n = O(n^{k/2})$

Hence $c_n = O(n^{k-1})$.

Using the so called "Rankin - Selberg" method the exponent $\frac{k}{2}$ in Thm 4.2 can be improved to $a_n = O(n^{\frac{k}{2}-\frac{1}{4}+\varepsilon})$ $\forall \varepsilon > 0$

Correct bound: Ramanujan conj'd (Deligne) is $O(n^{\frac{k}{2}-\frac{1}{2}+\varepsilon})$

Before we define the L-function attached to a modular form, we first look at Dirichlet series in general.

Defn (1) An ordinary Dirichlet series is a series of the shape

$$\sum_{n=1}^{\infty} a_n n^{-s} \ , \ a_n \in \mathbb{C} \ , \ s \in \mathbb{C}.$$

If we set all $a_n = 1$, we obtain the Riemann zeta function.

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$$

(2) A Dirichlet series $D(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ is called (absolutely) convergent if $\exists$ a complex number s_0 such that the series $\sum_{n=1}^{\infty} |a_n n^{-s_0}|$ converges in the usual sense

Set $s = \sigma + it$, $s_0 = \sigma_0 + it_0$

$$|n^{-s}| = |e^{-s\log n}| = |e^{-(\sigma+it)\log n}| = n^{-\sigma}$$
$$= n^{-\mathrm{Re}\, s}$$

and $n^{-\sigma} \leq n^{-\sigma_0}$ for $\sigma \geq \sigma_0$

Hence if a Dirichlet series converges abs at s_0, then it converges abs in the half-plane $\sigma \geq \sigma_0$

A half plane $\{s \in \mathbb{C} \mid \sigma > \tilde{\sigma}\}$ is called a half-plane of convergence if $D(s)$ converges abs. $\forall s$ in this half-plane

When $\tilde{\sigma} = -\infty$, half plane becomes all of $\mathbb{C}$

The union of all half-planes of convergence is the biggest of all half-planes of conv. and is called <u>the half-plane of abs. convergence</u>

Let $\{s \in \mathbb{C} \mid \sigma > \sigma_0\}$ be the half plane of abs. conv. for $D(s)$. Then $D(s)$ converges abs $\forall s$ with $\sigma > \sigma_0$ and it doesn't conv. abs for $\sigma < \sigma_0$

σ_0 is called the <u>abscissa of abs. convergence</u>

$D(s)$ represents an analytic function in the half-plane of abs. convergence

(See Apostol: Analytic number theory)

Example $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$, $\operatorname{Re} s > 1$

For the Riemann zeta function $\sigma_0 = 1$.

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If a sequence of numbers $a_1, a_2, a_n, \ldots$ has poly. growth of order N, ie $\exists\, C > 0$ s.t. $|a_n| < C n^N$ $\forall n$

Then $D(s) := \sum_{n=1}^{\infty} a_n n^{-s}$ has an abscissa of convergence $\sigma_0 \leq 1 + N$

$$|a_n n^{-s}| = C n^{-\sigma + N}$$

$$|D(s)| < C \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^{\sigma - N}}}_{\text{converges if } \sigma - N > 1}$$

ie $D(s)$ converges for $\sigma > 1 + N$. Hence abscissa of convergence $\sigma_0 \leq 1 + N$.

Lemma 4.4

If $f \in S_k$ then we have seen in Thm 4.2 that $a_n = O(n^{k/2})$ if $f = \sum a_n q^n$

Then the series $L(f, s) = \sum a_n n^{-s}$ converges abs. for $\operatorname{Re} s > k/2 + 1$.

We know that $\zeta(s)$ converges abs for $\operatorname{Re} s > 1$.

Before we go into the analytic properties of $\zeta(s)$ and $L(f, s)$, we mention also Dirichlet series associated Dirichlet characters mod N.

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Example 3) <u>Dirichlet L-functions</u>

<u>Defn</u> Let $N \geq 1$ be an integer

A Dirichlet character mod N is a map

$\chi : \mathbb{Z} \to \mathbb{C}$ s.t

1) $\chi(m+N) = \chi(m) \quad \forall m \in \mathbb{Z}$

2) $\chi(m) = 0 \iff \gcd(m,N) > 1$

3) $\forall\, m,n \in \mathbb{Z}$, $\chi(mn) = \chi(m)\chi(n)$

Recall that for any $a \in \mathbb{Z}$ with $\gcd(a,N)=1$ we have $a^{\phi(N)} = 1$

and $|(\mathbb{Z}/N\mathbb{Z})^*| = \phi(N)$

where ϕ is Euler ϕ-function.

Hence for any Dirichlet character mod N $\chi(a)$ is a $\phi(N)$-th root of unity

Characters mod N are in one-to-one correspondence with multiplicative characters of the abelian group $(\mathbb{Z}/N\mathbb{Z})^*$

Dirichlet introduced these characters in his study of the following question:

Let a, N be positive integers, s.t $\gcd(a,N)=1$

and consider the arithmetic progression

$a, a+N, a+2N, \ldots$

Does this sequence contain ∞ly many primes

For example: The primes of the form $1+k4$ are $5, 13, 17, 29, \ldots$

Primes of the form $4k+3$ are $3, 7, 11, 19, 23, 31, \ldots$

Dirichlet's theorem primes on arithmetic progressions state that there are ∞ly many primes for any arithmetic progression of the form $a, a+N, a+2N, \ldots$ with $\gcd(a, N) = 1$.

Stronger form in fact state that any such arithmetic progression $\sum_{p \equiv a (N)} \frac{1}{p}$ diverges

and different arithmetic progressions with the same modulus N have approximately the same number of primes

ie The primes are evenly distributed (asymptotically) among the $\phi(N)$ congruence classes modulo N.

eg. there are about the same # of primes of the form $4k+1$ and $4k+3$.

Dirichlet's thm says that if

$$\pi(x;4,a) = \#\{p \text{ prime} \mid p<x,\ p\equiv a \bmod 4\}$$

Then $\pi(x;4,1) \sim \pi(x;4,3) \sim \frac{1}{2}\frac{x}{\log x}$

where $\pi(x) = \#\{p \text{ prime} \mid p<x\} \sim \frac{x}{\log x}$ PNT.

That is half of all primes are of the form $4k+1$, and half $4k+3$.

One guesses that half of the time $\pi(x;4,1) > \pi(x;4,3)$ and half of the time $<$.

Numerical evidence suggest $\pi(x;4,3) > \pi(x;4,1)$ more frequently and is called Chebyshev's bias who observed it first time.

General case if $a, b < N$ with $\gcd(a,N) = (b,N) = 1$

a is quadratic residue mod N
b " non-residue mod N

Then $\pi(x;N,b) > \pi(x;N,a)$ occurs more often.

GRH $\Rightarrow$ Chebyshev's bias. known.

(see article of A. Granville, G. Martin "Prime number races" for an overview of this problem.

Dirichlet proved his theorem by showing the value of the Dirichlet L-function

$$L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \quad \text{of a non trivial character } \chi.$$
$$\mathrm{Re}\, s > 1$$

has in fact analy. continuation to $\mathrm{Re}\, s = 1$ and $L(\chi, 1) \neq 0$.

Rk ① Trivial character mod N is denoted by χ_0 and is the character s.t

$$\chi_0(a) = \begin{cases} 1 & \text{if } (a, N) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

② Since $(\chi(-1))^2 = \chi(-1)\chi(-1) = \chi(1) = 1$

$\chi(-1) = \pm 1$

A character χ mod N is called even if $\chi(-a) = \chi(a)$ and called odd if $\chi(-a) = -\chi(a)$ (ie $\chi(-1) = 1$, $\chi(-1) = -1$ resp)

③ We have orthogonality relation of Dirichlet characters.

ⓐ $$\sum_{a \bmod N} \chi(a) = \begin{cases} 0 & \text{if } \chi \neq \chi_0 \\ \phi(N) & \text{if } \chi = \chi_0 \end{cases}$$

ⓑ $$\sum_{\chi \bmod N} \chi(a) = \begin{cases} 0 & \text{if } a \not\equiv 1 \ (N) \\ \phi(N) & \text{if } a \equiv 1 \ (N) \end{cases}$$

sum is over all characters mod N.